JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **193**, 419–430 (1995)

The Unique Existence of Periodic Solutions of Linear Volterra Difference Equations

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Submitted by William F. Ames

Received April 4, 1994

The sufficient and necessary condition for the unique existence of periodic solutions of linear Volterra difference equations is obtained. Moreover, the approach taken in this work is different from the known results. Finally, the genericity of a relevant property is discussed. © 1995 Academic Press, Inc.

1. INTRODUCTION

For the linear Volterra difference equations of the form

$$\Delta y(n) = \sum_{j=0}^{\infty} K(j)\, y(n-j) + f(n), \qquad n \in Z, \tag{1}$$

where $y(n), f(n) \in R^k$, $K(n)$ is a $k \times k$ matrix for each $n \in Z$, Z is the integer set, and $\Delta y(n) = y(n+1) - y(n)$, the existence of periodic solutions have been investigated in [1, 2] where two different approaches have been adopted. It is assumed in [1] that the zero solution of the unperturbed equation of (1) is uniformly asymptotically stable; while the contraction principle is employed in [2] under a restriction on the size of the period.

Inspired by the ideas involved in [3] which deal with the linear Volterra differential equations, we use in this work a different approach to the known ones to establish the unique existence of periodic solutions for (1). Moreover, the obtained conditions are not only sufficient but also necessary. Incidentally, it will be pointed out that the main results, Theo-

*Supported by the National Natural Science Foundation of China.

0022-247X/95 $12.00

rems 3, 4, 5, 6, and 7 in [3] are not true by the arguments used there. In the end, we establish the genericity of the relevant property.

We note in (1) that y and f map Z into R^k. If $y = (y_1, y_2, \ldots, y_k)$ and $B = (b_{ij})_{k\times k}$, we define

$$|y| = \sum_{j=1}^{k} |y_j|, \qquad |B| = \sum_{i,j=1}^{k} |b_{ij}|.$$

For an integer $N > 0$ let $P_N = \{x : Z \to R^k \mid x(n+N) = x(n)\}$ and let

$$\|x\| = \max_{0\le s\le N-1} |x(s)| \qquad \text{for } x \in P_N. \tag{2}$$

Define the mean value of $x \in P_N$ by

$$m(x) = (1/N) \sum_{s=0}^{N-1} x(s).$$

Trivially, $m(x)$ is linear and continuous in x, and $|m(x)| \le \|x\|$.

Let $P_N^0 = \{x \in P_N \mid m(x) = 0\}$. It is clear that both P_N and P_N^0 are Banach spaces equipped with the norm defined by (2). For each $x \in P_N$, there corresponds an $\hat{x} \in P_N^0$ defined by

$$\hat{x}(n) = x(n) - m(x). \tag{3}$$

In what follows, we assume that $f \in P_N$, $\sum_{j=0}^{\infty} K(j)$ and $\sum_{j=0}^{\infty} |K(j)|$ are both convergent, and we let

$$\sum_{j=0}^{\infty} K(j) = \tilde{K}, \qquad \sum_{j=0}^{\infty} |K(j)| = \gamma \tag{4}$$

with some constant $k \times k$ martix $\tilde{K}$ and some constant $\gamma > 0$.

Define the operator A: $P_N \to P_N$ by

$$(Ax)(n) = \sum_{j=0}^{\infty} K(j)x(n-j). \tag{5}$$

Then A is a bounded linear operator with $\|A\| \le \gamma$.

It is easy to see that for any $x \in P_N$ we have

$$m(Ax) = Am(x) = \tilde{K}m(x). \tag{6}$$

Hence, $x \in P_N^0$ implies $Ax \in P_N^0$; i.e., A maps P_N^0 into P_N^0.

Finally, we define the operator $L: P_N^0 \to P_N^0$ by

$$(Lg)(n) = \Delta^{-1}g(n) - m(\Delta^{-1}g), \tag{7}$$

where $\Delta^{-1}g(n) = \sum_{i=0}^{n-1} g(i)$. Clearly, $\Delta^{-1}g \in P_N$ for $g \in P_N^0$.

In view of (5), we may rewrite (1) as

$$(\Delta - A)y = f. \tag{8}$$

We need the following lemmas to establish our main results.

LEMMA 1. *$L\Delta = I$, where I is the identity operator in* P_N^0.

Proof. If $G \in P_N^0$ and $\Delta G = g$, then $(L\Delta)G = L(\Delta G) = Lg = \Delta^{-1} g - m(\Delta^{-1}g)$. Since $\Delta((L\Delta)G) = \Delta(\Delta^{-1}g) - \Delta(m(\Delta^{-1}g)) = g$, we have $G - (L\Delta)G = c$, where $c \in R^k$ is constant.Thus, $c = m(c) = m(G) - m((L\Delta)G) = 0$, since $G \in P_N^0$ and $(L\Delta)G \in P_N^0$. Hence, $G = (L\Delta)G$ for each $G \in P_N^0$; i.e., $L\Delta = I$. Q.E.D.

LEMMA 2. $\Delta L = I$.

Proof. For each $g \in P_N^0$, we have $(\Delta L)g = \Delta(Lg) = \Delta(\Delta^{-1} g - m(\Delta^{-1} g)) = g$. This implies $\Delta L = I$. Q.E.D.

LEMMA 3. *If $y \in P_N$ is a solution of* (8), *then $\tilde{K}m(y) = -m(f)$.*

Proof. If $y \in P_N$ is a solution of (8); i.e., $\Delta y = Ay + f$, then by (6)

$$m(\Delta y) = m(Ay) + m(f) = \tilde{K}m(y) + m(f).$$

However,

$$m(\Delta y) = (1/N) \sum_{s=0}^{N-1} (\Delta y(s)) = (1/N)(y(N) - y(0)) = 0,$$

since $y \in P_N$. Hence, $\tilde{K}m(y) + m(f) = 0$. Q.E.D.

LEMMA 4. *Let* det $\tilde{K} \neq 0$. *If for each $g \in P_N^0$, the equation $(\Delta - A)x = g$ has a solution in P_N^0, then for any $f \in P_N$,* (8) *has a solution in P_N.*

Proof. Suppose that $f \in P_N$ and $x \in P_N^0$ is a solution of $(\Delta - A)x = \hat{f}$. Let $y = x - \tilde{K}^{-1}m(f)$. Then $\Delta y = \Delta x = Ax + \hat{f} = Ax + f - m(f) = Ay + f$. This means that y is a solution of (8). Clearly, $y \in P_N$. Q.E.D.

LEMMA 5. *Let* det $\tilde{K} = 0$ *and $m(f) \in \tilde{K}(R^k)$, where $\tilde{K}(R^k) = \{\tilde{K}x : x \in R^k\}$. If for any $g \in P_N^0$, the equation $(\Delta - A)x = g$ has a solution in P_N^0, then* (8) *has infinitely many solutions in P_N.*

Proof. Suppose that $f \in P_N$ and $x \in P_N^0$ is a solution of $(\Delta - A)x = \hat{f}$. Since det $\tilde{K} = 0$ and $m(f) \in \tilde{K}(R^k)$, there exist infinitely many $c \in R^k$ such that $\tilde{K}c = -m(f)$. However, for $y = x + c$ with $\tilde{K}c = -m(f)$ we have

$$\Delta y = \Delta x = Ax + \hat{f} = Ay + f.$$

Thus, (8) has infinitely many solutions in P_N. Q.E.D.

LEMMA 6. *If $m(f) \overline{\in} \tilde{K}(R^k)$, then* (8) *has no solutions in P_N.*

Proof. Suppose that $y \in P_N$ is a solution of (8). Then by Lemma 3 we know that $m(f) = \tilde{K}(-m(y)) \in \tilde{K}(R^k)$. This is a contradiction. Q.E.D.

Now let us consider the equation

$$(I - LA)x = h, \tag{9}$$

where $h \in P_N^0$.

LEMMA 7. *If for any $h \in P_N^0$,* (9) *has a solution in P_N^0, then so does* (8) *for any $f \in P_N^0$.*

Proof. Suppose that $f \in P_N^0$ and $y \in P_N^0$ is a solution of $(I - LA)x = Lf$. Then by Lemma 2,

$$\Delta(I - LA)y = \Delta(Lf) = f,$$

which implies

$$(\Delta - A)y = f,$$

since $\Delta(I - LA)y = \Delta y - \Delta(LAy) = (\Delta - A)y$. Hence, y is also a solution of (8). Q.E.D.

LEMMA 8. $\|L\| \leq (N - 1)/2$, *where L is the operator defined by* (7).

Proof. Obviously, by (7) L is a linear operator. For any $g \in P_N^0$ and $0 \leq s \leq N - 1$, we have

$$\begin{aligned}(Lg)(s) &= \sum_{i=0}^{s-1} g(i) - (1/N)\sum_{j=0}^{N-1}\sum_{i=0}^{j-1} g(i) = \sum_{i=0}^{s-1} g(i) - \sum_{i=0}^{N-2} ((N-i-1)/N)g(i) \\ &= \sum_{i=0}^{s-1} (1 - (N-i-1)/N)g(i) - \sum_{i=s}^{N-2} ((N-i-1)/N)g(i).\end{aligned}$$

Thus,

$$|(Lg)(s)| \le \|g\| \left(\sum_{i=0}^{s-1} (i+1)/N + \sum_{i=s}^{N-2} (N-i-1)/N \right)$$
$$= ((s^2 - sN + s)/N + (N-1)/2)\|g\| \qquad \text{for } 0 \le s \le N-1.$$

Noting that $s^2 - sN + s \le 0$ for $0 \le s \le N - 1$, we conclude that

$$|(Lg)(s)| \le ((N-1)/2)\|g\| \qquad \text{for } 0 \le s \le N-1$$

and thus $\|L\| \le (N-1)/2$. Q.E.D.

Remark 1. Incidentally, we note that there is a mistake in [3] as well as in the cited reference [2] in [3], in which there should be $\|L\| \le T/2$ rather than $T/4$.

By Lemma 8 we know that L is a linear bounded operator. Furthermore, we have the following.

LEMMA 9. *The operator* $LA\colon P_N^0 \to P_N^0$ *is compact.*

Proof. Since A maps P_N^0 into P_N^0 and $L\colon P_N^0 \to P_N^0$, clearly, $LA\colon P_N^0 \to P_N^0$. The fact that both A and L are linear and bounded implies that LA is also linear and bounded and thus continuous. Moreover, LA maps any bounded set in P_N^0 into a bounded set in P_N^0 which is relatively compact in P_N^0, since P_N^0 is a closed subspace of R^{kN}. Therefore, LA is a compact operator. Q.E.D.

The following corollary follows from Lemma 9 and the Riesz–Szauder theorem (cf. [4]).

COROLLARY 1 *$\sigma(LA)$ is a finite set or a countable set, where $\sigma(LA)$ denotes the spectrum of the operator LA.*

LEMMA 10 (cf. [4]). *If B is a Banach space and $T\colon B \to B$ is a bounded linear operator, then the spectral radius*

$$r(T) = \sup_{\lambda \in \sigma(T)} |\lambda| = \lim_{n\to\infty} \sqrt[n]{\|T^n\|}.$$

2. MAIN RESULTS

With the aid of the lemmas in Section 1, we are now in a position to establish the following results.

THEOREM 1. *If* $1 \bar{\in} \sigma(LA)$, *then* (8) (*i.e.*, (1)) *has a unique solution for each* $f \in P_N$ *if and only if* $\det \tilde{K} \neq 0$.

Proof. If $1 \bar{\in} \sigma(LA)$, then the operator $I - LA\colon P_N^0 \to P_N^0$ is a regular operator. Thus, for each $h \in P_N^0$, (9) has a unique solution, $x = (I - LA)^{-1}h \in P_N^0$. It now follows from Lemma 7 that (8) has a solution in P_N^0 for each $f \in P_N^0$. If $\det \tilde{K} \neq 0$, then by Lemma 4, for each $f \in P_N$, (8) has a solution $y \in P_N$.

Suppose that (8) has another solution $z \in P_N$ for the same $f \in P_N$. Then it is easily seen that $\hat{y}$ and $\hat{z}$ both are solutions of

$$(\Delta - A)x = \hat{f}.$$

In fact, let $(\Delta - A)y = f$, then by (6) and Lemma 3 we have

$$(\Delta - A)\hat{y} = (\Delta - A)y - (\Delta - A)m(y) = f + Am(y) = \hat{f}.$$

Therefore,

$$L(\Delta - A)\hat{y} = L\hat{f}, \qquad L(\Delta - A)\hat{z} = L\hat{f}.$$

By Lemma 1 it follows that

$$\hat{y} - LA\hat{y} = L\hat{f}, \qquad \hat{z} - LA\hat{z} = L\hat{f}$$

and, thus,

$$(I - LA)(\hat{y} - \hat{z}) = 0.$$

This implies

$$\hat{y} - \hat{z} = 0,$$

since $I - LA$ is regular.

Hence, by (3) we have for some $c \in R^k$, $y - z = c$. But then, since both y and z are solutions of (8), we obtain

$$(\Delta - A)c = (\Delta - A)(y - z) = 0,$$

or

$$-Ac = 0; \qquad \text{i.e., } \tilde{K}c = 0.$$

Noting that $\det \tilde{K} \neq 0$, we get $c = 0$ and, thus, $y = z$. Therefore, (8) has a unique solution in P_N for each $f \in P_N$.

Conversely, if $\det \tilde{K} = 0$, then there exist infinitely many $c \in R^k$ such that $\tilde{K}c = 0$. For a given $f \in P_N$, once (8) has an N-periodic solution y, then obviously $y + c$ with $\tilde{K}c = 0$ is also an N-periodic solution of (8). Hence, (8) cannot have a unique N-periodic solution. Q.E.D.

COROLLARY 2. If $N < 1 + 2/\gamma$, *then* (8) *has a unique N-periodic solution for each* $f \in P_N$ *if and only if* $\det \tilde{K} \neq 0$.

Proof. By Lemmas 10 and 8 we have $r(LA) \leq \|LA\| \leq (N - 1)\gamma/2$. Hence, if $N < 1 + 2/\gamma$, then $r(LA) < 1$. This implies that $1 \overline{\in} \sigma(LA)$. Thus, applying Theorem 1 yields the desired conclusion. Q.E.D.

Remark 2. Trivially, Corollary 2 is an improvement of Theorem 1 in [2]. Moreover, the arguments given here are without using any fixed point theorems, including the contraction principle.

To go one step further, from Theorem 1 we can show the following result.

THEOREM 2. *Let* $1 \overline{\in} \sigma(LA)$ *and* $\det \tilde{K} = 0$. *Then*

(i) *if* $m(f) \in \tilde{K}(R^k)$, (8) *has infinitely many N-periodic solutions;*

(ii) *if* $m(f) \overline{\in} \tilde{K}(R^k)$, (8) *has no N-periodic solutions.*

Proof. Since $1 \overline{\in} \sigma(LA)$, as seen from above, for each $f \in P_N$, the equation $(I - LA)x = L\hat{f}$ has a solution $y \in P_N^0$. (Note that $f \in P_N$ implies that $\hat{f} \in P_N^0$ and, thus, $L\hat{f} \in P_N^0$.) Thus, by Lemma 2,

$$(\Delta - A)y = \Delta(I - LA)y = \Delta L\hat{f} = \hat{f}.$$

(i) If $m(f) \in \tilde{K}(R^k)$, there exist infinitely many $c \in R^k$ such that $\tilde{K}c = -m(f)$ since $\det \tilde{K} = 0$. As pointed out in the proof of Lemma 5, $y + c$ with $\tilde{K}c = -m(f)$ is an N-periodic solution of (8). Hence, (8) has infinitely many N-periodic solutions.

(ii) Follows immediately from Lemma 6. Q.E.D.

THEOREM 3. *If* $1 \in \sigma(LA)$, *then* (8) *has either infinitely many N-periodic solutions or no periodic solutions.*

Proof. Since $1 \in \sigma(LA)$ and P_N^0 is a finite dimensional space, 1 must be an eigenvalue of LA. Hence, there are infinitely many $g \in P_N^0$, $g \neq 0$, such that $(I - LA)g = 0$, and, thus, $(\Delta - A)g = 0$.

Then if (8) has a solution $y \in P_N$, it is easily seen that $y + g$ with $(I - LA)g = 0$ is also a solution of (8). Clearly, $y + g \in P_N$. Q.E.D.

Remark 3. It should be pointed out that the proof of Theorem 3 in [3] is wrong. Because in an infinite dimensional space such as P_T^0, $1 \in \sigma(LA)$ does not imply that 1 is an eigenvalue of LA (cf. [4]). Thus, it is not true that there are infinitely many $g \in P_T^0$, $g \neq 0$, such that $(I - LA)g = 0$. It is reasonable to suspect that Theorem 3 in [3] is not valid.

By virtue of Theorems 1 and 3 we obtain the following.

COROLLARY 3. *Equation* (8) (*i.e.*, (1)) *has a unique N-periodic solution for each* $f \in P_N$ *if and only if* $1 \overline{\in} \sigma(LA)$ *and* $\det \tilde{K} \neq 0$.

Furthermore, we have the following result.

THEOREM 4. *The restriction of the operator* $(\Delta - A)^{-1}: P_N \to P_N$ *to* P_N^0 *is linear and bounded if and only if* $1 \overline{\in} \sigma(LA)$ *and* $\det \tilde{K} \neq 0$. *If* $1 \overline{\in} \sigma(LA)$ *and* $\det \tilde{K} \neq 0$, *then* $(\Delta - A)^{-1} = \sum_{i=0}^{\infty} (LA)^i L$ *holds on* P_N^0.

Proof. First, we note that $\Delta - A$ maps P_N into P_N and is a bounded linear operator. Trivially, it follows from Corollary 3 that $(\Delta - A)^{-1}$ does not make sense if $1 \in \sigma(LA)$ or $\det \tilde{K} = 0$.

Conversely, if $1 \overline{\in} \sigma(LA)$ and $\det \tilde{K} \neq$, 0, then by Corollary 3, $(\Delta - A)^{-1}$ is well defined on P_N. For any $f \in P_N^0 \subset P_N$, let $y = (\Delta - A)^{-1} f$; i.e., $(\Delta - A) y = f$. This implies that $(I - LA) y = Lf$. Since $1 \overline{\in} \sigma(LA)$, $I - LA$ is a regular operator. Hence, $y = (I - LA)^{-1}(Lf)$. Therefore, $(\Delta - A)^{-1} f = ((I - LA)^{-1} L) f$ holds for all $f \in P_N^0$. This shows that there holds $(\Delta - A)^{-1} = (I - LA)^{-1} L$ on P_N^0.

Since both $(I - LA)^{-1}$ and L are linear and bounded operators on P_N^0, the restriction of $(\Delta - A)^{-1}$ to P_N^0 is linear and bounded. Moreover,

$$(\Delta - A)^{-1} = (I - LA)^{-1} L = \sum_{i=0}^{\infty} (LA)^i L \qquad \text{on } P_N^0. \qquad \text{Q.E.D.}$$

Remark 4. The arguments in the proof of Theorem 5 in [3] have some problems, since the domain of $(D - A)^{-1}$ is P_T, while the domain of $(I - LA)^{-1}L$ is P_T^0. (Note that for $g \in P_T$, Lg is even not in P_T.) $(D - A)^{-1} f = (I - LA)^{-1} Lf$ holds only for $f \in P_T^0$; it cannot be concluded that $(D - A)^{-1} = (I - LA)^{-1} L$. Hence, the statements of Theorem 5 in [3] are doubtful.

COROLLARY 4. *If* $N < 1 + 2/\gamma$ *and* $\det \tilde{K} \neq 0$, *then the restriction of* $(\Delta - A)^{-1}$ *to* P_N^0 *is a bounded linear operator, and*

$$\|(\Delta - A)^{-1}\| \leq (N - 1)/(2 - (N - 1)\gamma) \qquad \textit{on } P_N^0.$$

Proof. Noting that $N < 1 + 2/\gamma$ implies $1 \overline{\in} \sigma(LA)$, we derive from Theorem 4 that

$$\begin{aligned} \|(\Delta - A)^{-1}\| &\leq \sum_{i=0}^{\infty} \|LA\|^i \|L\| \leq ((N - 1)/2) \sum_{i=0}^{\infty} ((N - 1)\gamma/2)^i \\ &= (N - 1)/(2 - (N - 1)\gamma) \qquad \text{on } P_N^0. \qquad \text{Q.E.D.} \end{aligned}$$

EXAMPLE. Consider the scalar equation

$$\Delta y(n) = ay(n) + \sum_{j=1}^{\infty} b(1/2)^j y(n-j) + f(n), \qquad n \in Z, \tag{10}$$

where $y \in R$; a, b are constant; $f \in P_N$; $N > 0$ is an integer. Then

$$K(0) = a, \qquad K(j) = b(1/2)^j, \qquad j = 1, 2, \ldots, \qquad \bar{K} = \sum_{j=0}^{\infty} K(j) = a + b,$$

and

$$\gamma = \sum_{j=0}^{\infty} |K(j)| = |a| + |b|.$$

Assume that $N < 1 + 2/(|a| + |b|)$. It then follows from Corollary 2 and Theorem 2 that

(i) if $a + b \neq 0$, then (10) has a unique N-periodic solution;
(ii) if $a + b = 0$ and $m(f) = 0$; i.e., $f \in P_N^0$, then (10) has infinitely many N-periodic solutions;
(iii) if $a + b = 0$ and $m(f) \neq 0$; i.e., $f \overline{\in} P_N$, then (10) has no N-periodic solutions.

3. FURTHER INVESTIGATION

Let W be the set of all sequences $\{K(j)\}$ of $m \times m$ matrices such that $\sum_{j=0}^{\infty} |K(j)| < \infty$. Define the norm $\|K\| = \sum_{j=0}^{\infty} |K(j)|$ for $K \in W$.

Obviously, $(W, \|\cdot\|)$ is a Banach space. If the distance in W is, as usual, defined by $d(x, y) = \|x - y\|$, then W is a complete metric space.

Let $M_K = \sum_{j=0}^{\infty} K(j)$ and define the operator A_K: $P_N \to P_N$ by

$$(A_K g)(n) = \sum_{j=0}^{\infty} K(j) g(n-j) \qquad \text{for } g \in P_N.$$

Let

$$Q = \{K \in W: \det M_K \neq 0 \text{ and } 1 \overline{\in} \sigma(LA_K)\},$$

where L is defined as in (7). We introduce the following definitions.

DEFINITION 1. Let B be a complete metric space. A property $\mathcal{P}$ on the elements of B is said to be generic in B if $\mathcal{P}$ is shared by an open and dense subset of B.

DEFINITION 2. A sequence of matrices $K = \{K(j)\} \in W$ is said to have property $\mathcal{P}_N$ if Eq. (1) has a unique N-periodic solution for each $f \in P_N$.

To assert the genericity of property $\mathcal{P}_N$ we need the following lemmas.

LEMMA 11. *The set Q is open in W.*

Proof. Let $K_0 \in Q$. Then $\det M_{K_0} \neq 0$ and $1 \,\overline{\in}\, \sigma(LA_{K_0})$. Since

$$|M_K - M_{K_0}| = \left|\sum_{j=0}^{\infty} (K(j) - K_0(j))\right| \leq \sum_{j=0}^{\infty} |K(j) - K_0(j)| = \|K - K_0\|,$$

there exists a sufficiently small $\delta_1 > 0$ such that if $\|K - K_0\| < \delta_1$ then the difference between each element of M_K and its corresponding element in M_{K_0} can be arbitrarily small. Hence, $\det M_{K_0} \neq 0$ implies $\det M_K \neq 0$, provided $\|K - K_0\| < \delta_1$.

On the other hand, since $1 \,\overline{\in}\, \sigma(LA_{K_0})$, i.e., $I - LA_{K_0}$ is a regular operator on P_N^0, there exists a $b > 0$ such that $\|(I - LA_{K_0})x\| \geqslant b$ for $x \in P_N^0$ with $\|x\| = 1$. Noting that

$$\|(I - LA_K) - (I - LA_{K_0})\| \leq \|L\| \, \|A_K - A_{K_0}\| \leq \|L\| \, \|K - K_0\|,$$

we have

$$\begin{aligned} \|(I - LA_{K_0})x\| - \|(I - LA_K)x\| &\leq \|(I - LA_K)x - (I - LA_{K_0})x\| \\ &\leq \|L\| \, \|K - K_0\| \, \|x\| = \|L\| \, \|K - K_0\| \qquad \text{for } x \in P_N^0 \text{ with } \|x\| = 1. \end{aligned}$$

Choose a $\delta_2 > 0$ with $\delta_2\|L\| < b/2$. Then if $\|K - K_0\| < \delta_2$ we have

$$\|(I - LA_K)x\| \geqslant \|(I - LA_{K_0})x\| - \|L\| \, \|K - K_0\| > b - b/2 = b/2 \tag{11}$$

for $x \in P_N^0$ with $\|x\| = 1$.

Suppose that $1 \in \sigma(LA_K)$. Then 1 must be an eigenvalue of LA_K, since P_N^0 is a finite-dimensional space. It follows that there exists an $x^* \neq 0$, $x^* \in P_N^0$, with $(I - LA_K)x^* = 0$, or $(I - LA_K)(x^*/\|x^*\|) = 0$. This contradicts (11). Thus $1 \,\overline{\in}\, \sigma(LA_K)$ if $\|K - K_0\| < \delta_2$.

Therefore, $K \in Q$ if $\|K - K_0\| < \min\{\delta_1, \delta_2\}$. This shows that Q is open. Q.E.D.

LEMMA 12. *The set Q is dense in W.*

Proof. Let $Q_1 = \{K \in W: \det M_K \neq 0\}$ and $Q_2 = \{K \in W: 1 \overline{\in} \sigma(LA_K)\}$. Then $Q = Q_1 \cap Q_2$. To claim that Q is dense in W it suffices to show that for each $K_0 \in W - Q$ there exists a sequence $\{K_n\} \subset Q$ such that $\|K_n - K_0\| \to 0$ as $n \to \infty$.

Note that it is easy to verify

$$W - Q = (W - Q_1) \cup ((W - Q_2) \cap Q_1). \tag{12}$$

Let $K_0 \overline{\in} Q_1$. Then $\det M_{K_0} = 0$ and there are nonsingular $k \times k$ matrices P and R such that

$$PM_{K_0}R = \begin{pmatrix} I_i & O_{i,k-i} \\ O_{k-i,i} & O_{k-i,k-i} \end{pmatrix}, \qquad 0 \le i < k,$$

where I_i denotes that $i \times i$ identity matrix and $O_{i,j}$ denotes the $i \times j$ matrix with zero elements. Let

$$S = P^{-1}\begin{pmatrix} O_{i,i} & O_{i,k-i} \\ O_{k-i,i} & I_{k-i} \end{pmatrix} R^{-1}, \qquad K^*(j) = \left(\frac{1}{2}\right)^{j+1} S.$$

Then $M_{K^*} = \sum_{j=0}^{\infty} (\frac{1}{2})^{j+1} S = S$. Let $K_\varepsilon = K_0 + \varepsilon K^*$ for $\varepsilon \neq 0$ and, thus,

$$\|K_\varepsilon - K_0\| = \|\varepsilon K^*\| = |\varepsilon|\, \|K^*\| = |\varepsilon|\, |S|. \tag{13}$$

Noting that $M_{K_\varepsilon} = M_{K_0} + \varepsilon M_{K^*} = M_{K_0} + \varepsilon S$, we obtain

$$PM_{K_\varepsilon}R = \begin{pmatrix} I_i & O_{i,k-i} \\ O_{k-i,i} & O_{k-i,k-i} \end{pmatrix} + \varepsilon \begin{pmatrix} O_{i,i} & O_{i,k-i} \\ O_{k-i,i} & I_{k-i} \end{pmatrix} \equiv T$$

and, thus, $M_{K_\varepsilon} = P^{-1}TR^{-1}$. It follows that $\det M_{K_\varepsilon} \neq 0$, i.e., $K_\varepsilon \in Q_1$, since P^{-1}, R^{-1} are nonsingular and $\det T = \varepsilon^{k-i} \neq 0$.

On the other hand, since LA_K is a bounded linear operator on the finite dimensional space P_N^0, under a chosen basis of P_N^0, each LA_K corresponds to a matrix denoted by J_K, which is called the matrix of LA_K. Correspondingly, the matrix of operator $I - LA_K$ is $I - J_K$.

Now no matter $K_0 \in Q_2$ or $K_0 \overline{\in} Q_2$, i.e., $I - LA_{K_0}$ is regular or not, we may choose sufficiently small $\varepsilon > 0$ so that $\det (I - J_{K_\varepsilon}) = \det (I - J_{K_0} - \varepsilon J_K^*) \neq 0$ and, thus, $I - LA_{K_\varepsilon}$ is a regular operator, i.e., $1 \overline{\in} \sigma(LA_{K_\varepsilon})$, since P_N^0 is finite dimensional. Hence, $K_\varepsilon \in Q_2$ for small $\varepsilon > 0$.

Therefore, $K_\varepsilon \in Q_1 \cap Q_2 = Q$ for small $\varepsilon > 0$, and it then follows from (13) that $\|K_\varepsilon - K_0\| \to 0$ as $\varepsilon \to 0$.

Now let $K_0 \overline{\in} Q_2$ and $K_0 \in Q_1$, then $1 \in \sigma(LA_{K_0})$ and $\det M_{K_0} \neq 0$. Let $K_\varepsilon = (1 + \varepsilon)K_0$. Suppose there exists some $\delta > 0$ such that for any ε with $|\varepsilon| < \delta$ we have $1 \in \sigma(LA_{K_\varepsilon})$. Then $LA_{K_\varepsilon} = (1 + \varepsilon)LA_{K_0}$, which implies that $(1 + \varepsilon)^{-1} \in \sigma(LA_{K_0})$ for $|\varepsilon| < \delta$. However, LA_{K_0} is compact; it leads to a contradiction by Corollary 1. Therefore, there must exist $\varepsilon_n : 0 < \varepsilon_n < 1/n$ with $1 \overline{\in} \sigma(LA_{K_{\varepsilon_n}})$; i.e., $K_{\varepsilon_n} \in Q_2$. On the other hand, since $\det M_{K_{\varepsilon_n}} = \det((1 + \varepsilon_n)M_{K_0}) = (1 + \varepsilon_n)^k \det M_{K_0} \neq 0$, we know that $K_{\varepsilon_n} \in Q_1$ for such ε_n. Hence, $\{K_{\varepsilon_n}\} \subset Q_1 \cap Q_2 = Q$. It is easy to see that

$$\|K_{\varepsilon_n} - K_0\| = |\varepsilon_n| \, \|K_0\| < \|K_0\|/n \to 0 \qquad \text{as } n \to \infty.$$

Thus, by (12) we conclude that Q is dense in W. Q.E.D.

Remark 5. In the arguments deriving the conclusion "Q is dense in W" in [3], the author implicitly used the statement "the intersection of two dense sets in W is dense in W." It seems to us that this is not always true. Hence, Lemma 11 and Theorems 6 and 7 in [3] are not conceivable to us.

By Corollary 3 and Lemmas 11 and 12 we finally obtain the following:

THEOREM 5. *The property $\mathcal{P}_N$ is generic in W.*

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